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Hierarchies of integrable equations associated with hyperelliptic Lie algebras

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Abstract
Using a family of special quasigraded Lie algebras on hyperelliptic curves we construct new hierarchies of integrable nonlinear equations admitting zero-curvature representations. We show that in the case of the rational degeneration of the curve they coincide with Heisenberg magnet hierarchies.

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1. Introduction

During the last few decades several mathematical schemes that enable one to find hierarchies of nonlinear integrable equations were proposed. Most of them have algebraic origin and are based on the so-called zero-curvature equations [1]

$$\frac{\partial U(x,t_k,w)}{\partial t_k} - \frac{\partial V_k(x,t_k,w)}{\partial x} + [U(x,t_k,w), V_k(x,t_k,w)] = 0 \tag{1}$$

where U, V_k are the matrix-valued functions, depending on the dynamical variables, their derivatives and an additional complex parameter w usually called 'spectral'. Historically, zero-curvature equations appeared as compatibility conditions for the set of auxiliary linear problems [1]

$$\frac{\partial \vec{\psi}}{\partial x} = U(x,t_k,w)\vec{\psi} \tag{2}$$

$$\frac{\partial \vec{\psi}}{\partial t_k} = V_k(x,t_k,w)\vec{\psi}. \tag{3}$$

In the case of the rational dependence of U and V_k on w, there are several possible Lie-theoretical interpretations of equations (1). The first and the most known interpretation is based on the so-called functional Hamiltonian formalism [1, 2] which permits one to interpret equations (1) as a Hamiltonian equation of the Euler–Arnold type on the orbits of the centrally extended algebra of periodic functions of variable x with the values in $\mathfrak{g} \otimes \mathrm{Pol}(w, w^{-1})$.

The second Lie-theoretical approach to zero-curvature equations is based thoroughly on the Lie algebra $\mathfrak{g}\otimes \mathrm{Pol}(w, w^{-1})$. It was independently discovered in the papers [3–5]. It treats equations (1) as a compatibility condition for the infinite set of commuting Hamiltonian flows on the Lie algebras $\widetilde{\mathfrak{g}}_+ \equiv \mathfrak{g}\otimes \mathrm{Pol}(w)$ or $\widetilde{\mathfrak{g}}_- \equiv \mathfrak{g}\otimes \mathrm{Pol}(w^{-1})$

$$\frac{\partial L(w)}{\partial t_l} = [\nabla H_l(L(w)), L(w)] \tag{4}$$

$$\frac{\partial L(w)}{\partial t_k} = [\nabla H_k(L(w)), L(w)] \tag{5}$$

(here $L(w)$ belongs to $\widetilde{\mathfrak{g}}_\pm$ or their quotient algebras). Commuting Hamiltonians $\{H_k(L)\}$ are constructed using the Kostant–Adler scheme. Compatibility conditions for the corresponding Hamiltonian flows read as

$$\frac{\partial \nabla H_l(L(w))}{\partial t_k} - \frac{\partial \nabla H_k(L(w))}{\partial t_l} + [\nabla H_l(L(w)), \nabla H_k(L(w))] = 0. \tag{6}$$

We see that no 'space' variable x is *a priori* singled out in this approach. One may put $t_0 \equiv x, \nabla H_0(L(w)) \equiv U(x, w), \nabla H_k(L(w)) \equiv V_k(x, w), k > 0$ in order to obtain zero-curvature equations in the form (1). The most important ingredients in both Lie theoretical schemes are loop algebras and their decompositions into the sum of two subalgebras (Kostant–Adler scheme).

In [5–7], in order to obtain integrable hierarchies with elliptic dependence of the spectral parameters, more complicated infinite-dimensional Lie algebras, namely special $so(3)$-valued algebras of meromorphic functions on elliptic curves were proposed. These algebras admit decompositions into the sum of two subalgebras, that enable one to find an infinite set of commuting Hamiltonians and write down zero-curvature equations with elliptic spectral parameters as compatibility conditions for the set of corresponding Hamiltonian flows.

In [8, 9] the algebraic construction of [7] was generalized on the matrix Lie algebras of the higher ranks and higher genus algebraic curves. As a result quasigraded Lie algebras of special $so(d)$, $sp(d)$ and $gl(d)$-valued meromorphic functions on the hyperelliptic curves were obtained. They were shown to admit the Kostant–Adler scheme. In such a way new finite-dimensional integrable Hamiltonian systems of the Euler–Arnold type were constructed [8, 9].

The main goal of the present paper is to extend the 'second' Lie-theoretical approach to the case of the above mentioned special hyperelliptic Lie algebras $\widetilde{\mathfrak{g}}_{\mathcal{H}}$, where $\mathfrak{g}$ is equal to $so(d)$, $sp(d)$ or $gl(d)$. We show that the compatibility condition of the two Hamiltonian flows on the Lie algebras $\widetilde{\mathfrak{g}}^{\pm}_{\mathcal{H}}$ or their quotient algebras also leads to the zero-curvature equations. In such a way we obtain new hierarchies of integrable Hamiltonian equations admitting zero-curvature representations. We show that simplest equations of the hierarchy coincide with a kind of multiparametric deformation of the generalized Heizenberg magnet equations, where the parameters of the deformations are the branching points of the hyperelliptic curve. We consider also some examples of the higher equations from the hierarchy.

The structure of the present paper is as follows: in section 2 we introduce algebras $\tilde{\mathfrak{g}}_{\mathcal{H}}$ and describe their properties, in section 3 we describe integrable Hamiltonian systems on the quotient algebras of the subalgebras $\tilde{\mathfrak{g}}^{\pm}_{\mathcal{H}}$, and in section 4 we construct new hierarchies of integrable equations admitting zero-curvature representations.

2. Kostant–Adler admissible Lie algebras on hyperelliptic curves

Hyperelliptic curve embedded in $\mathbb{C}^d$. Let us consider in the space $\mathbb{C}^d$ with the coordinates $w_1, w_2, \ldots, w_d$ the following system of quadrics:

$$w_i^2 - w_j^2 = a_j - a_i \qquad i, j = 1, d \tag{7}$$

where a_i are arbitrary complex numbers. The rank of this system is $d-1$, so the substitution

$$w_i^2 = w - a_i \qquad y = \prod_{i=1}^{d} w_i \qquad y^2 = \prod_{i=1}^{d}(w - a_i)$$

solves these equations and defines the equation of the hyperelliptic curve $\mathcal{H}$. Hence equations (7) define embedding of the hyperelliptic curve $\mathcal{H}$ in the linear space $\mathbb{C}^d$.

Classical Lie algebras. Let $\mathfrak{g}$ denote one of the classical matrix Lie algebras $gl(d)$, $so(d)$ and $sp(d)$ over the field of the complex numbers. We will need the explicit form of their bases. Let $I_{i,j} \in \mathrm{Mat}(d, C)$ be a matrix defined as

$$(I_{ij})_{ab} = \delta_{ia}\delta_{jb}.$$

Evidently, a basis in the algebra $gl(d)$ could be built from the matrices $X_{ij} \equiv I_{ij}, i, j \in 1, \ldots, d$. The commutation relations in $gl(d)$ will have the standard form

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j}.$$

The basis in the algebra $so(d)$ could be chosen as: $X_{ij} \equiv I_{ij} - I_{i,j}, i, j \in 1, \ldots, d$, with 'skew-symmetry' property $X_{ij} = -X_{ji}$ and the following commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j} + \delta_{j,l} X_{k,i} - \delta_{k,i} X_{j,l}.$$

The basis in the algebra $sp(n)$ we choose as $X_{ij} = I_{ij} - \epsilon_i \epsilon_j I_{-i,-j}, |i|, |j| \in 1, \ldots, d$, with the property $X_{i,j} = -\epsilon_i \epsilon_j X_{-j,-i}$, where $\epsilon_j = \operatorname{sign} j$ and commutation relations

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j} + \epsilon_i \epsilon_j (\delta_{j,-l} X_{k,-i} - \delta_{k,-i} X_{-j,l}).$$

Algebras on the curve. For the basic elements X_{ij} of all three algebras $gl(d)$, $so(d)$ and $sp(d)$ and arbitrary $n \in \mathbb{Z}$ we introduce the following algebra-valued functions on the curve $\mathcal{H}$, or to be more precise on its double covering

$$X_{ij}^n = X_{ij} \otimes w^n w_i w_j.$$

(Here we put $w_{-i} \equiv w_i$ in the case of $sp(d)$.) The next theorem holds true.

Theorem 1.

(i) Elements X_{ij}^n form $n \in \mathbb{Z}$ quasigraded Lie algebra $\tilde{\mathfrak{g}}_{\mathcal{H}}$ with the following commutation relations:

(1) $$\left[X_{ij}^n, X_{kl}^m\right] = \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} \quad \text{for the } gl(d) \tag{8a}$$

(2) $$\begin{aligned}\left[X_{ij}^n, X_{kl}^m\right] = {} & \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + \delta_{jl} X_{ki}^{n+m+1} - \delta_{ik} X_{jl}^{n+m+1} \\ & + a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} + a_i \delta_{ik} X_{jl}^{n+m} - a_j \delta_{jl} X_{ki}^{n+m} \\ & \text{for the } so(d)\end{aligned} \tag{8b}$$

(3) $$\begin{aligned}\left[X_{ij}^n, X_{kl}^m\right] = {} & \delta_{kj} X_{il}^{n+m+1} - \delta_{il} X_{kj}^{n+m+1} + \epsilon_i \epsilon_j (\delta_{j-l} X_{k-i}^{n+m+1} - \delta_{i-k} X_{j-l}^{n+m+1}) \\ & + a_i \delta_{il} X_{kj}^{n+m} - a_j \delta_{kj} X_{il}^{n+m} + a_i \epsilon_i \epsilon_j (a_i \delta_{i-k} X_{j-l}^{n+m} \\ & - a_j \delta_{j-l} X_{k-i}^{n+m}) \qquad \text{for the } sp(d).\end{aligned} \tag{8c}$$

(ii) Algebra $\tilde{\mathfrak{g}}_{\mathcal{H}}$ as a linear space admits a decomposition into the direct sum of two subalgebras: $\tilde{\mathfrak{g}}_{\mathcal{H}} = \tilde{\mathfrak{g}}_{\mathcal{H}}^+ + \tilde{\mathfrak{g}}_{\mathcal{H}}^-$, where subalgebras $\tilde{\mathfrak{g}}_{\mathcal{H}}^+$ and $\tilde{\mathfrak{g}}_{\mathcal{H}}^-$ are generated by the elements X_{ij}^0, and X_{ij}^{-1} respectively.

Coadjoint representation and its invariants. In order to define Hamiltonian systems on $\tilde{\mathfrak{g}}^*_{\mathcal{H}}$ first we have to define dual space $\tilde{\mathfrak{g}}^*_{\mathcal{H}}$. Subsequently it will be convenient to identify $\tilde{\mathfrak{g}}^*_{\mathcal{H}}$ with $\tilde{\mathfrak{g}}_{\mathcal{H}}$. For this purpose we will use the pairing between $\tilde{\mathfrak{g}}_{\mathcal{H}}$ and $\tilde{\mathfrak{g}}^*_{\mathcal{H}}$ defined as follows: for $L(w) \in \tilde{\mathfrak{g}}^*_{\mathcal{H}}$ and $X(w) \in \tilde{\mathfrak{g}}_{\mathcal{H}}$ we have

$$\langle X(w), L(w)\rangle_f = \oint_{w=0} f(w) y^{-2}(w)(X(w) \mid L(w)) \tag{9}$$

where $f(w)$ is the arbitrary holomorphic function of the spectral parameter w. It will be convenient to take $f(w) = w^{-(K+1)}$. We will denote these pairings by $\langle\, ,\rangle_K$. Under this choice, the pairing Lax operator will have the following form:

$$L(w) = \sum_{m\in Z} \sum_{i,j=1}^{d} l_{ij}^{(m)} \frac{w^{m-1} y^2(w)}{w_i w_j} X^*_{ij} \equiv \sum_{m\in Z} \sum_{i,j=1}^{d} l_{ij}^{(m)} Y_{ij}^m \tag{10}$$

where $l_{ij}^{(m)}$ are coordinate functions on the dual space, which will be our basic dynamical variables.

From the explicit form of pairing (9) and dual space (10) follows the next proposition.

Proposition 1.

(i) *The action of the algebra $\tilde{\mathfrak{g}}_{\mathcal{H}}$ on its dual space $\tilde{\mathfrak{g}}^*_{\mathcal{H}}$ coincides with commutator:*

$$ad^*_{X(w)} L(w) = [L(w), X(w)]. \tag{11}$$

(ii) *Functions $H^k_m(L(w)) = \oint_{w=0} w^{-m-1} \operatorname{tr} L(w)^k$, where $m \in \mathbb{Z}$, are invariants of the coadjoint representation.*

Conclusion: Lie algebras $\tilde{\mathfrak{g}}_{\mathcal{H}}$ admit a decomposition into the direct sum of two subalgebras and possess an infinite number of invariant functions. Hence, they could be used to construct integrable Hamiltonian systems via the Kostant–Adler scheme.

3. Hamiltonian systems via quasigraded algebras

In this section we will construct Hamiltonian systems on the coadjoint orbits of the Lie groups that correspond to algebras $\tilde{\mathfrak{g}}^{\pm}_{\mathcal{H}}$, admitting Lax pair representations with hyperelliptic spectral parameter and possessing a lot of mutually commuting integrals of motion. To define Hamiltonian systems on the coadjoint orbits we have to define first the Lie–Poisson structures and their Lie–Poisson subspaces.

3.1. Lie–Poisson structures and Lie–Poisson subspaces

Lie–Poisson structures. In the space $\tilde{\mathfrak{g}}^*_{\mathcal{H}}$ one can define many Lie–Poisson structures using different pairings. We will use pairings $\langle\, ,\ \rangle_K$. They define brackets on $P(\tilde{\mathfrak{g}}^*_{\mathcal{H}})$ in the following way:

$$\{F(L), G(L)\}_K = \sum_{n,m\in Z} \sum_{i,j,k,l=1}^{d} \left\langle L(w), \left[X_{ij}^{-n+K}, X_{kl}^{-m+K}\right]\right\rangle_K \frac{\partial G}{\partial l_{ij}^{(n)}} \frac{\partial F}{\partial l_{kl}^{(m)}}. \tag{12}$$

From proposition 1 follows the next statement:

Proposition 2. *Functions $H^k_m(L(w))$ are central for brackets $\{\ ,\ \}_K$.*

Let us explicitly calculate Poisson brackets (12). It is easy to show that for the coordinate functions $l_{ij}^{(m)}$ these brackets will have the following form:

$$(1)\quad \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\}_K = \delta_{kj} l_{il}^{(n+m-K-1)} - \delta_{il} l_{kj}^{(n+m-K-1)} + a_i \delta_{il} l_{kj}^{(n+m-K)} - a_j \delta_{kj} l_{il}^{(n+m-K)} \quad \text{for } gl(d) \tag{13a}$$

$$(2)\quad \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\}_K = \delta_{kj} l_{il}^{(n+m-K-1)} - \delta_{il} l_{kj}^{(n+m-K-1)} + \delta_{jl} l_{ki}^{(n+m-K-1)} - \delta_{ik} l_{jl}^{(n+m-K-1)} + a_i \delta_{il} l_{kj}^{(n+m-K)} - a_j \delta_{kj} l_{il}^{(n+m-K)} + a_i \delta_{ik} l_{jl}^{(n+m-K)} - a_j \delta_{jl} l_{ki}^{(n+m-K)} \quad \text{for } so(d) \tag{13b}$$

$$(3)\quad \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\}_K = \delta_{kj} l_{il}^{(n+m-K-1)} - \delta_{il} l_{kj}^{(n+m-K-1)} + \epsilon_i \epsilon_j \left(\delta_{j-l} l_{k-i}^{(n+m-K-1)} - \delta_{i-k} l_{j-l}^{(n+m-K-1)}\right) + a_i \delta_{il} l_{kj}^{(n+m-K)} - a_j \delta_{kj} l_{il}^{(n+m-K)} + \epsilon_i \epsilon_j \left(a_i \delta_{i-k} l_{j-l}^{(n+m-K)} - a_j \delta_{j-l} l_{k-i}^{(n+m-K)}\right) \quad \text{for } sp(d). \tag{13c}$$

Lie–Poisson subspaces. Let us now consider the following subspace $\mathcal{M}_N \subset \tilde{\mathfrak{g}}_{\mathcal{H}}^*$:

$$\mathcal{M}_N = \sum_{m=1}^{N+1} (\tilde{\mathfrak{g}}_{\mathcal{H}}^*)_m \qquad \text{where} \quad (\tilde{\mathfrak{g}}_{\mathcal{H}}^*)_m = \sum_{ij} l_{ij}^{(m)} Y_{ij}^m.$$

For the each fixed N we will be interested in the brackets $\{\ ,\ \}_K$ where $K = 0$ or $K = N + 1$. The following proposition holds true:

Proposition 3. *Brackets* $\{\ ,\ \}_0$ *and* $\{\ ,\ \}_{N+1}$ *could be restricted onto the space* $\mathcal{M}_N$.

Proof. From the explicit form of the Poisson brackets (13) it follows that subspaces $\mathcal{M}_{1,\infty} = \sum_{m=1}^{\infty} (\tilde{\mathfrak{g}}_{\mathcal{H}}^*)_m$ and $\mathcal{M}_{-\infty,N} = \sum_{m=-\infty}^{N+1} (\tilde{\mathfrak{g}}_{\mathcal{H}}^*)_m$ are Lie–Poisson subalgebras with respect to the brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_{N+1}$, respectively (it is not difficult to show that these Poisson algebras are isomorphic to $\tilde{\mathfrak{g}}_{\mathcal{H}}^-$ and $\tilde{\mathfrak{g}}_{\mathcal{H}}^+$). From the explicit form of the Lie–Poisson brackets it also follows that for any integer p subspaces $\mathcal{J}_{p,\infty} = \sum_{m=p}^{\infty} (\tilde{\mathfrak{g}}^*{}_{\mathcal{H}})_m$ and $\mathcal{J}_{-\infty,p} = \sum_{-\infty}^{p} (\tilde{\mathfrak{g}}_{\mathcal{H}}^*)_m$ are ideals in the above described Lie–Poisson subalgebras. That proves the proposition. □

3.2. Commutative subalgebras and Lax equations

Commutative subalgebras. We start this subsection with the following theorem:

Theorem 2. *Let functions* $\left\{H_m^k(L)\right\}$ *be defined as in proposition 1. They generate commutative algebra with respect to the restriction of the brackets* $\{\ ,\ \}_0$ *and* $\{\ ,\ \}_{N+1}$ *onto* $\mathcal{M}_N$.

Proof. From the explicit form of the Poisson brackets (13) it follows that the subspaces complementary in $\tilde{\mathfrak{g}}_{\mathcal{H}}^*$ to $\mathcal{M}_{1,\infty} = \sum_{m=1}^{\infty} (\tilde{\mathfrak{g}}_{\mathcal{H}}^*)_m$ and $\mathcal{M}_{-\infty,N} = \sum_{m=-\infty}^{N+1} (\tilde{\mathfrak{g}}_{\mathcal{H}}^*)_m$ are also Lie–Poisson subalgebras (with respect to the corresponding brackets $\{\ ,\ \}_0$ and $\{\ ,\ \}_{N+1}$). Hence, from the Kostant–Adler scheme it follows that restriction of the invariant functions $H_m^k(L)$ onto the subalgebras $(\mathcal{M}_{1,\infty},\ \{\ ,\ \}_0)$ and $(\mathcal{M}_{-\infty,N},\ \{\ ,\ \}_{N+1})$ forms commutative subalgebra with respect to the corresponding brackets. Now to prove the theorem it is enough to take into consideration that subspace $(\mathcal{M}_N,\ \{\ ,\ \}_0)$ and $(\mathcal{M}_N,\ \{\ ,\ \}_{N+1})$ coincide with the quotient algebras $\mathcal{M}_{-\infty,N}/\mathcal{J}_{-\infty,0}$ or $\mathcal{M}_{1,\infty}/\mathcal{J}_{N+2,\infty}$, and the projection on the quotient algebra is the canonical homomorphism.

That proves the theorem. □

Previous theorem gives us a large, commutative—with respect to both Lie–Poisson brackets on $\mathcal{M}_N$—subalgebra generated by the functions $\{H^k_m(L)\}$. We will need the explicit form of some of them.

Example 1. Second-order integrals $\{H^2_s\}$. Let

$$H^2(L(w)) \equiv \tfrac{1}{2}\mathrm{tr}\, L(w)^2 = \sum_{s=0}^{D} H^2_s(L) w^s$$

where $D = 2d + 2N - 2$.

Higher order in the parameter w terms of the above expansions could be shown to be the following:

$$H^2_{D-1} = \sum_{i,j=1}^{d} l^{(N+1)}_{ij} l^{(N)}_{ji} - \frac{1}{2}\sum_{i,j=1}^{d}\sum_{k=1}^{d}(2a_k - a_i - a_j) l^{(N+1)}_{ij} l^{(N+1)}_{ji}$$

$$H^2_D = \frac{1}{2}\sum_{i,j=1}^{d} l^{(N+1)}_{ij} l^{(N+1)}_{ji}.$$

It is necessary to emphasize that among functions $\{H^k_m(L)\}$ there are 'geometric invariants'—Casimir functions of the Lie–Poisson brackets $\{\,,\}_0$ or $\{\,,\}_{N+1}$ and commuting integrals, that generate nontrivial flows on the corresponding coadjoint orbits. The following theorem enables one to distinguish Casimir functions from the nontrivial integrals:

Theorem 3. *Let us consider functions $\{H^k_s(L)\}$ restricted to the subspace $\mathcal{M}_N$. Then*

(i) *functions $H^k_s(L)$ are Casimir functions of the brackets $\{\,,\}_0$ if $(k-1)N + k(d-1) \leqslant s \leqslant kN + k(d-1)$ for $\mathfrak{g} = gl(d)$, $\mathfrak{g} = sp(d)$ and $\mathfrak{g} = so(d)$ for $k = 2r$.*

(ii) *functions $H^k_s(L)$ are Casimir functions of the brackets $\{\,,\}_{N+1}$ if $0 \leqslant s \leqslant N$ for $\mathfrak{g} = gl(d)$, $\mathfrak{g} = sp(d)$ and $\mathfrak{g} = so(d)$ for $k = 2r$.*

(iii) *for the arbitrary coordinate function $l^{(n)}_{ij}$ the following identity holds true: $\left\{l^{(n)}_{ij}, H^k_s\right\}_0 = \left\{H^k_{s+N+1}, l^{(n)}_{ij}\right\}_{N+1}$.*

Proof. Let $\tilde{X}^m_{ij}, \tilde{Y}^m_{ij}$ be vector fields, that correspond to the brackets $\{\,,\}_0$ and $\{\,,\}_{N+1}$, i.e.:

$$\tilde{X}^m_{ij} F(L) \equiv \left\{l^{(m)}_{ij}, F(L)\right\}_0, \qquad \tilde{Y}^m_{ij} F(L) \equiv \left\{l^{(m)}_{ij}, F(L)\right\}_{N+1}.$$

Proof of the theorem will be based on the following proposition:

Proposition 4. *Let $L = \sum_{i,j=1}^{d} L_{ij}(w) X_{ij}$, where $L_{ij}(w) = y^2(w)\sum_{m=1}^{N+1} w_i^{-1} w_j^{-1} l^{(m)}_{ij} w^{m-1}$. The following identity holds true:*

$$\left(w^l \tilde{X}^l_{ij} + w^{l-N-1}\tilde{Y}^l_{ij}\right) F(L(w)) = (w_i w_j)^{-1}\sum_{m,n=1}^{d} C^{(mn)}_{(ij),(kl)} L_{mn}(w)\frac{\partial F(L(w))}{\partial L_{kl}(w)}$$

where $C^{(mn)}_{(ij),(kl)}$ are structure constants of the algebras $gl(d)$, $sp(d)$ or $so(d)$.

Proposition is proved by direct calculations. Now, to prove the theorem it is enough to take into consideration that $H^n(L) \equiv \mathrm{tr}\, L^n(w)$ is a Casimir function, hence:

$$\sum_{m,n=1}^{d} C^{(mn)}_{(ij),(kl)} L_{mn}(w)\frac{\partial H^n(L(w))}{\partial L_{kl}(w)} \equiv 0$$

to expand $H^n(L(w))$ in the power series in the spectral parameter w and to compare the summands with the equal degrees of w. That proves the theorem. □

Lax equations. Let us consider Hamiltonian equations of the form

$$\frac{\mathrm{d}l_{ij}^{(k)}}{\mathrm{d}t} = \left\{ l_{ij}^{(k)}, H(l_{kl}^{(m)}) \right\}_K \tag{14}$$

where Hamiltonian H is one of the functions $\left\{H_s^k\right\}$ and $\{\,,\}_K$ are the brackets $\{\,,\}_0$ or $\{\,,\}_{N+1}$ restricted to the subspace $\mathcal{M}_N$.

These equations could be written in the Lax form

$$\frac{\mathrm{d}L(w)}{\mathrm{d}t} = [L(w), M(w)] \tag{15}$$

where $L(w) \in \mathcal{M}_{s,p}$, and a second operator is defined as follows: $M(w) = \nabla_K H(L(w))$. Here

$$\nabla_K H(L(w)) = \sum_{m=1}^{N+1} \sum_{ij=1}^{d} \frac{\partial H}{\partial l_{ij}^{(m)}} X_{ij}^{-m+K} \tag{16}$$

is an algebra-valued gradient of H, and $K = 0$ or $K = N + 1$.

From part (iii) of theorem 3 follows the next important corollary.

Corollary 1. *Hamiltonian equations for the Hamiltonian H_s^m and the brackets $\{\,,\}_0$ coincide with the Hamiltonian equations for the Hamiltonian H_{s+N+1}^m and brackets $\{\,,\}_{N+1}$:*

$$\frac{\partial l_{ij}^{(n)}}{\partial t} = \left\{ l_{ij}^{(n)}, H_s^m\left(l_{kl}^{(m)}\right) \right\}_0 = \left\{ H_{s+N+1}^m\left(l_{kl}^{(m)}\right), l_{ij}^{(n)} \right\}_{N+1}. \tag{17}$$

Remark 1. In the Lax-pair form the statement of the corollary could be written as

$$\frac{\partial L(w)}{\partial t} = \left[L(w), \nabla_0 H_s^k(L(w))\right] = \left[\nabla_{N+1} H_{s+N+1}^k(L(w)), L(w)\right]. \tag{18}$$

4. Hierarchies of integrable evolutionary equations

In this section we will discuss the so-called 'finite-gap extension method' that enables one to construct non-linear evolutionary equations starting from the finite-dimensional Hamiltonian systems.

4.1. Zero-curvature equations as the compatibility conditions

Usually zero-curvature equations are considered as the compatibility conditions of the set of auxiliary linear problems. Another natural interpretation is to consider zero-curvature equations as the compatibility condition of the set of Hamiltonian flows. We will develop this approach starting from the general situation.

Case of the general algebras. Let us consider some simple (or reductive) Lie algebra $\mathfrak{g}$. Let $\tilde{\mathfrak{g}}$ be the algebra of the functions of one complex variable (we do not specify their properties here) with the values in $\mathfrak{g}$, or some of its quotient algebra. We will underline the functional character of the algebra $\tilde{\mathfrak{g}}$ denoting its elements as $X(w)$. Let $L(w) \in \tilde{\mathfrak{g}}^*$ be the generic element of the dual space. It has the following form:

$$L(w) = \sum_k \sum_{a=1}^{\dim\mathfrak{g}} l_a^{(k)} X_a^{k*} \equiv \sum_{a=1}^{\dim\mathfrak{g}} l_a(w) X_a$$

where X_a is a basic element of $\mathfrak{g}$, $l_a^{(k)}$ is the coordinate function on $\tilde{\mathfrak{g}}^*$ and X_a^{k*} is a basic element of the $\tilde{\mathfrak{g}}^*$. Let us assume that with respect to some pairing $\langle\ ,\ \rangle$ we have that $\tilde{\mathfrak{g}}^*$ coincide with $\tilde{\mathfrak{g}}$ as linear spaces. Under such assumptions the coadjoint representation of $\tilde{\mathfrak{g}}$ coincides with commutator.

Let us assume that we possess two Hamiltonian equations of Euler–Arnold type on the Lie algebra $\tilde{\mathfrak{g}}$. Due to the assumptions given above, these Hamiltonian equations could be written in the Lax form

$$\frac{\partial L(w)}{\partial t_1} = [\nabla h_1(L(w)), L(w)] \qquad \frac{\partial L(w)}{\partial t_2} = [\nabla h_2(L(w)), L(w)] \tag{19}$$

where $\nabla h_1(L(w))$ is a $\tilde{\mathfrak{g}}$-valued gradient of $h_1(L(w))$, defined via the pairing $\langle\ ,\ \rangle$.

Our considerations will be based on the following proposition:

Proposition 5. *Let Hamiltonians $h_1(L(w))$ and $h_2(L(w))$ commute with respect to the Lie–Poisson brackets on $\tilde{\mathfrak{g}}^*$. Then algebra-valued gradients of the functions h_1 and h_2 satisfy the following equations:*

$$\frac{\partial \nabla h_1(L)}{\partial t_2} - \frac{\partial \nabla h_2(L)}{\partial t_1} - [\nabla h_2(L), \nabla h_1(L)] = \mathrm{d}C(L) \tag{20}$$

where $\mathrm{d}C(L) = \sum_{a=1}^{\dim \mathfrak{g}} \frac{\partial C(L(w))}{\partial l_a(w)} X_a$ *and C(L) is some $\mathfrak{g}$-invariant function.*

Proof. Hamiltonians $h_1(L)$ and $h_2(L)$ commute with respect to the Lie–Poisson brackets on $\mathfrak{g}^*$ That means that corresponding Hamiltonian flows also commute. That is why we have

$$\frac{\partial^2 L}{\partial t_2 \partial t_1} = \frac{\partial^2 L}{\partial t_1 \partial t_2}.$$

Using the equation of motion (19) we obtain the following differential equations:

$$\frac{\partial}{\partial t_2}([\nabla h_1(L), L]) = \frac{\partial}{\partial t_1}([\nabla h_2(L), L]).$$

They are equivalent to the zero-curvature-type equations. Indeed, using the Leibnitz rule, the Hamiltonian differential equations (19) and Jacobi identity in $\mathfrak{g}$ we obtain

$$\left[\left(\frac{\partial \nabla h_1(L)}{\partial t_2} - \frac{\partial \nabla h_2(L)}{\partial t_1} - [\nabla h_2(L), \nabla h_1(L)]\right), L\right] = 0.$$

It is equivalent to the following 'generalized zero-curvature equations':

$$\frac{\partial \nabla h_1(L)}{\partial t_2} - \frac{\partial \nabla h_2(L)}{\partial t_1} - [\nabla h_2(L), \nabla h_1(L)] = \mathrm{d}C(L)$$

where $C(L)$ is some $\mathfrak{g}$-invariant function[1]. Indeed, due to the fact that $C(L(w))$ is an invariant function we have: $[\mathrm{d}C(L(w)), L(w)] = 0$. This proves the proposition. □

Remark 2. Proposition 5 by itself does not provide the integrability of the corresponding differential equations in the partial derivatives on the elements $L(t_1, t_2, w)$ (20). Indeed, equations (20) do not coincide with the zero-curvature equations. However, in special cases, when the algebra $\tilde{\mathfrak{g}}$ is graded or quasigraded, and the elements $\nabla h_i(L)$ possess the fixed quasigrade, it is possible to show that the Casimir function $C(L(w))$ should be taken as constant. This will be illustrated in the next subsection on the example of the algebras $\tilde{\mathfrak{g}}^{\pm}_{\mathcal{H}}$.

[1] Rigorously speaking, we have to take here $p(w)C(L)$ instead of $C(L)$, where $p(w)$ is some function of the spectral parameter w. But this will not influence our subsequent considerations and we will put $p \equiv$ const.

Case of Kostant–Adler admissible hyperelliptic Lie algebras. Let us now consider the case of $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}^{\pm}_{\mathcal{H}}$ or their quotient algebras. Let us consider the Hamiltonian equations on their dual space. In detail, let us consider two non-trivial Hamiltonian equations on their Poisson subspace $\mathcal{M}_N$ generated by two commuting functions $h_{s_1} \equiv H^2_{s_1},\ h_{s_2} \equiv H^2_{s_2}$, where $s_1 = D-(N+k+1)$, $s_2 = D-(N+l+1)$ and the brackets $\{\ ,\ \}_0$ or, equivalently (see theorem 3) by the commuting functions $h_{D-k} \equiv H^2_{D-k},\ h_{D-l} \equiv H^2_{D-l}$, where $k, l \in 0, 2d+N-3$ and the brackets $\{\ ,\ \}_{N+1}$. They could be written in the Lax-pair form:

$$\frac{\partial L(w)}{\partial t_k} = \left[\nabla_{N+1} H^2_{D-k}(L(w)), L(w)\right] \qquad \frac{\partial L(w)}{\partial t_l} = \left[\nabla_{N+1} H^2_{D-l}(L(w)), L(w)\right]. \tag{21}$$

The following proposition holds true:

Proposition 6. *Matrix-valued functions $\nabla_{N+1} h_{D-k}$ and $\nabla_{N+1} h_{D-l}$ satisfy zero-curvature equations:*

$$\frac{\partial \nabla_{N+1} h_{D-k}}{\partial t_l} - \frac{\partial \nabla_{N+1} h_{D-l}}{\partial t_k} + [\nabla_{N+1} h_{D-k}, \nabla_{N+1} h_{D-l}] = 0. \tag{22}$$

Proof. The statement of this proposition is a consequence of proposition 5. Indeed due to this proposition we obtain

$$\frac{\partial \nabla_{N+1} h_{D-k}}{\partial t_l} - \frac{\partial \nabla_{N+1} h_{D-l}}{\partial t_k} + [\nabla_{N+1} h_{D-k}, \nabla_{N+1} h_{D-l}] = \mathrm{d}C(L).$$

On the other hand $\nabla_{N+1} h_{D-k}$ and $\nabla_{N+1} h_{D-l}$ are the elements of the algebra $\tilde{\mathfrak{g}}^{+}_{\mathcal{H}}$ that lie in the 'strip' of the fixed grade $\mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^k$ and $\mathfrak{g}^0 + \mathfrak{g}^1 + \cdots + \mathfrak{g}^l$ respectively. Due to the fact that $\tilde{\mathfrak{g}}^{+}_{\mathcal{H}}$ is a quasigraded Lie algebra, $[\nabla_{N+1} h_{D-k}, \nabla_{N+1} h_{D-l}]$ lie in the strip $\mathfrak{g}^0 + \mathfrak{g}^1 + + \cdots + \mathfrak{g}^{k+l+1}$. On the other hand it is easy to prove that for each fixed N and for each invariant function $C_r(L)$ of the order r element $\mathrm{d}C_r(L)$, $r \geqslant 2$ lies in the strip $\sum_{k=0}^{r(N+d-1)} \mathfrak{g}^k$. Hence, at least for $k, l \leqslant N+d-2$, we should put $C(L) \equiv$ const $\Rightarrow \mathrm{d}C(L) = 0$. This proves the proposition. □

Remark 3. The previous proposition provides us with a large class of $U-V$ pairs admitting zero-curvature equations. The last are non-linear equations in the partial derivatives on the dynamical variables—matrix elements of the matrices $L(w)$. It is easy to see that the space $\mathcal{M}_N$ could be viewed as a finite-gap sector of these equations (see [10] for the general definition of the finite-gap sectors of equations admitting zero-curvature representations). Due to the fact that in each space $\mathcal{M}_N$ zero-curvature equations are the results of the commutativity of Hamiltonian flows from the set of commuting functions constituting the finite-dimensional integrable Hamiltonian system, we will say that the resulting equations in the partial derivatives are 'integrable in the finite-gap sense'. In the next sections we will consider several examples of such equations.

4.2. Integrable deformation of Heisenberg magnet equations

Now let us consider the simplest integrable equations that could be obtained from the results of the previous subsection. For this purpose we will consider the case $k = 0$, $l = 1$. From the previous section it follows that matrix-valued functions

$$U(x, t, w) \equiv \nabla_{N+1} h_D \qquad V_1(x, t, w) \equiv \nabla_{N+1} h_{D-1}$$

where $x \equiv t_0$, $t \equiv t_1$, constitute a new type of $U-V$ pair, satisfying zero-curvature equations. Direct calculation gives their explicit form

$$V_1 = \sum_{i,j=1}^{d} \left(l_{ij}^{(N)} + (a_i + a_j) l_{ij}^{(N+1)} \right) w_i w_j X_{ij} + w \sum_{i,j=1}^{d} l_{ij}^{(N+1)} w_i w_j X_{ij}$$

$$U = \sum_{i,j=1}^{d} l_{ij}^{(N+1)} w_i w_j X_{ij}.$$

Here we have imposed the following constraints on the parameters a_i: $\sum_{k=1}^{d} a_k = 0$.

Remark 4. In the case of rational degeneration we obtain the $U-V$ pair for the Heisenberg magnet (HM) equations

$$U = w \sum_{i,j=1}^{d} l_{ij}^{(N+1)} X_{ij}, \qquad V_1 = \sum_{i,j=1}^{d} w l_{ij}^{(N)} X_{ij} + w^2 \sum_{i,j=1}^{d} l_{ij}^{(N+1)} X_{ij}.$$

Let us consider the obtained zero-curvature equations in more detail. For this purpose, let us introduce auxiliary notation. Let us introduce two new spectral parameter-independent matrices, depending on the dynamical variables $l_{ij}^{(N)}, l_{ij}^{(N+1)}$

$$L = \sum_{i,j=1}^{d} l_{ij}^{(N+1)} X_{ij} \qquad M_1 = \sum_{i,j=1}^{d} \left(l_{ij}^{(N)} + (a_i + a_j) l_{ij}^{(N+1)} \right) X_{ij}. \tag{23}$$

By direct calculations one can prove the following proposition:

Proposition 7. *Equations* (22) *for* $k = 0$, $l = 1$ *are equivalent to the following system of differential equations on the matrices* M_1 *and* L:

$$\frac{\partial L}{\partial t} - \frac{\partial M_1}{\partial x} = [L, M_1]_A \tag{24a}$$

$$\frac{\partial L}{\partial x} = [L, M_1]. \tag{24b}$$

Here $[L, M_1]_A \equiv L A M_1 - M_1 A L$, *and* $A = diag(a_1, a_2, \ldots, a_d)$.

Remark 5. Note that equations (24) have the same form in each 'finite-gap' sector $\mathcal{M}_N$, i.e. effectively they do not depend on the number N.

Equations (24) contain two types of dynamical variables—matrices L and M_1. Nevertheless, in some cases, using equation (24b) along with special G-invariant constraints on matrix L, it is possible to express M_1 explicitly via L and its derivatives, which leads to nonlinear equations in the partial derivatives on the matrix elements of matrix L. We will call them *deformed Heisenberg magnet (DHM) equations*. Let us find explicit form of these equations for different Lie algebras $\mathfrak{g}$ and different forms of G-invariant constraints on $L \in \mathfrak{g}$.

Case of $\mathfrak{g} = so(3)$. In this case we can consider elements of $\mathfrak{g}$ as vectors:

$$\vec{L} = \sum_{k=1}^{3} L_k X_k \qquad \vec{M}_1 = \sum_{k=1}^{3} M_k X_k$$

where $X_k \equiv \epsilon_{ijk} X_{ij}$, and rewrite equations (24) as follows:

$$\frac{\partial \vec{L}}{\partial t} - \frac{\partial \vec{M}_1}{\partial x} = A[\vec{L} \times \vec{M}_1] \tag{25a}$$

$$\frac{\partial \vec{L}}{\partial x} = [\vec{L} \times \vec{M}_1]. \tag{25b}$$

Here $\vec{L} \times \vec{M}$ denotes a vector product of two vectors

$$\vec{L} \times \vec{M}_1 = \sum_{i,j,k=1}^{3} \epsilon_{ijk} L_i M_j X_k$$

$A(\vec{L})$ denotes the action of the matrix A on vector $\vec{L}$: $A\vec{L} \equiv \sum_{k=1}^{3} a_k L_k X_k$.

As in the case of the Heisenberg magnet and Landau–Lifshitz equations we will impose the following constraint on the vector $\vec{L}$:

$$\vec{L}^2 = \sum_{i=1}^{3} L_i^2 = 1.$$

This means that vector L belongs to the coadjoint orbit of $SO(3)$—sphere S^2. The invariance of this constraint with respect to all commuting flows follows from the 'finite-gap' considerations. Indeed, equations (25) could be viewed as compatibility conditions of two Hamiltonian flows on $\mathcal{M}_N$ with respect to the brackets $\{\,,\}_0$. On the other hand, as follows from theorem 1, $H_D^2 \equiv \vec{L}^2$ is a Casimir function for these brackets for arbitrary N. That proves the invariance of the above constraint.

Using this constraint we can express $\vec{M}_1$ via $\vec{L}$. Indeed, it is easy to prove that for vectors $\vec{L}$, satisfying the above constraint, the substitution

$$\vec{M}_1 = -\vec{L} \times \frac{\partial \vec{L}}{\partial x} + c_1(L)\vec{L}$$

gives a general solution of equations (25b). In order to obtain the explicit form of the coefficient $c_1(L)$ it is enough to take into consideration that in each 'finite-gap' sector $\mathcal{M}_N$ matrices M_1 and L are expressed via $l_{ij}^{(N+1)}$ and $l_{ij}^{(N)}$ and

$$H_{D-1}^2 = (\vec{M}_1, \vec{L}) + \tfrac{1}{2}(A\vec{L}, \vec{L})$$

is a Casimir function of the brackets $\{\,,\}_0$, and hence is equal to constant $\mathcal{M}_N$ for each N. (Here by $(\vec{L}, \vec{M})$ we denoted the scalar product of vectors $\vec{L}$ and $\vec{M}_1$.)

Rescaling the time variable $t \to -t$ we obtain the following equations on vector $\vec{L}$:

$$\frac{\partial \vec{L}}{\partial t} = \vec{L} \times \frac{\partial^2 \vec{L}}{\partial x^2} - A\frac{\partial \vec{L}}{\partial x} + \frac{\partial}{\partial x}\left(\left(\frac{1}{2}(A\vec{L}, \vec{L}) - c_1\right)\vec{L}\right) \tag{26}$$

In such a form these equations were discovered in [7].

Case of $\mathfrak{g} = gl(d)$. Let us consider degenerate coadjoint orbits of the group $G = Gl(d)$ (we assume that the basic field is $\mathbb{C}$ or $\mathbb{R}$) in the space of the matrices L, given by the matrix equations

$$L^2 = \alpha L$$

where α is an arbitrary complex or real number. It is easy to show that these equations determine the union of the $GL(d)$ coadjoint orbits of the type $GL(d)/Gl(d_1) \times GL(d_2)$. Proof of the consistency of these equations with equations (24) could be done by returning to the spaces $\mathcal{M}_N$. Indeed, equations (24) are the compatibility conditions for equations (21). The last could be interpreted as a pair of Hamiltonian equations written with respect to the brackets $\{\,,\}_0$. On the other hand, taking into consideration that $L \equiv \sum_{i,j=1}^{d} l_{ij}^{(N+1)} X_{ij}$, it is easy to prove that $F_{ij} \equiv (L^2)_{ij} - \alpha(L)_{ij}$ are covariants with respect to the brackets $\{\,,\}_0$ (see (13)). Hence equations $F_{ij} = 0$ determine an invariant manifold—union of the coadjoint orbits of the group $Gl(d)$ of the type described above.

Now, to obtain the DHM equation we have to express the matrix M_1 via the matrix L. It is not difficult to show that if $L^2 = \alpha L$, then $\left[L, \left[L, \frac{\partial L}{\partial x}\right]\right] = \alpha^2 \frac{\partial L}{\partial x}$ and substitution

$$M_1 = (\alpha^2)^{-1}\left[L, \frac{\partial L}{\partial x}\right] + c_1(L)L$$

give the general solution of equations (24b). An explicit expression for $c_1(L)$ could be obtained, using the fact that in each 'finite-gap' sector $\mathcal{M}_N$ matrices M_1 and L are expressed via $l_{ij}^{(N+1)}$ and $l_{ij}^{(N)}$. Taking this into consideration we obtain that

$$H_{D-1}^2 = \mathrm{tr}(LM_1) - \mathrm{tr}(AL^2)$$

is a Casimir function for the brackets $\{\,,\}_0$ in each sector $\mathcal{M}_N$. Using this constraint on the matrices M and L, we obtain for $c_1(L)$ the next expression

$$c_1(L) = (2\,\mathrm{tr}(L^2))^{-1}\,\mathrm{tr}(AL^2) + c_1 = (2\mathrm{tr}L)^{-1}\,\mathrm{tr}(AL) + c_1.$$

Note, that $\mathrm{tr}\,L^2 = \alpha\,\mathrm{tr}\,L = \mathrm{const}$. Using the obtained expression for M_1 and $c_1(L)$ and rescaling the time variable: $t \to \alpha^{-2}t$, we obtain that equations (24a) on the described above orbits are equivalent to one nonlinear equation of the following form:

$$\frac{\partial L}{\partial t} = \left[L, \frac{\partial^2 L}{\partial x^2}\right] + \left[L, \left[L, \frac{\partial L}{\partial x}\right]\right]_A + \alpha^2 \frac{\partial}{\partial x}(c_1(L)L). \tag{27}$$

Case of $\mathfrak{g} = so(2n)$ or $\mathfrak{g} = sp(n)$. Let us consider the special degenerate coadjoint orbit of $G = SO(2d)$ or $G = SP(d)$ given by the matrix equations:

$$L^2 = \beta 1.$$

These are the orbits of the type $SO\,(2d)/GL\,(d)$ and $SP\,(d)/GL\,(d)$ and β is a complex or real number that determines the initial point of the orbit. In analogy to the previous case it is possible to show that on these orbits equations (24) are equivalent to one nonlinear equation in the partial derivatives of the following form:

$$\frac{\partial L}{\partial t} = \left[L, \frac{\partial^2 L}{\partial x^2}\right] + \left[L, \left[L, \frac{\partial L}{\partial x}\right]\right]_A + \beta c_1 \frac{\partial L}{\partial x} \tag{28}$$

where c_1 is an arbitrary constant.

Remark 6. The difference in the forms of the resulting equations (27) and (28) is the result of the different forms of the equations of the corresponding coadjoint orbits.

Remark 7. Note that in the process of deriving the final form of nonlinear evolutionary equations (26)–(28) we have used two Lie–Poisson brackets in the auxiliary Poisson subspace $\mathcal{M}_N$. Brackets $\{\,,\}_{N+1}$ were used to generate $U-V$ pairs for the zero-curvature equations, and brackets $\{\,,\}_0$ for obtaining invariant constraints on our dynamical variables.

4.3. Higher equations from hierarchy

In this subsection we will consider other integrable equations, admitting zero-curva-ture representations, that could be obtained using the approach described above. For this purpose we will consider zero-curvature conditions of the type (22), with $k = 0, l > 1$:

$$\frac{\partial \nabla_{N+1} h_D}{\partial t_l} - \frac{\partial \nabla_{N+1} h_{D-l}}{\partial x} + [\nabla_{N+1} h_D, \nabla_{N+1} h_{D-l}] = 0. \tag{29}$$

We will use the notation $U = \nabla_{N+1} h_D$, $V_l = \nabla_{N+1} h_{D-l}$. Direct calculation gives

$$U = \sum_{i,j=1}^{d} l_{ij}^{(N+1)} w_i w_j X_{ij} \qquad V_l = \sum_{k=0}^{l} \sum_{i,j=1}^{d} m^{(k)}\left(l_{ij}^{(N+1)}, l_{ij}^{(N)}, \ldots, l_{ij}^{(N+1-k)}\right) w^k w_i w_j X_{ij}$$

where $m^{(k)}\left(l_{ij}^{(N+1)}, l_{ij}^{(N)}, \ldots, l_{ij}^{(N+1-k)}\right)$ depend on the deformation parameters a_i, a_j and are linear in the dynamical variables $l_{ij}^{(N-k)}$. For example, we have

$$m^{(0)}\left(l_{ij}^{(N+1)}\right) \equiv l_{ij}^{(N+1)} \qquad m^{(1)}\left(l_{ij}^{(N+1)}\right) \equiv l_{ij}^{(N)} + (a_i + a_j)l_{ij}^{(N+1)}.$$

Remark 8. In the rational degeneration, when $a_i = 0$, for $i = 1, d$ we obtain

$$m^{(k)}\left(l_{ij}^{(N+1)}, l_{ij}^{(N)}, \ldots, l_{ij}^{(N+1-k)}\right) \equiv l_{ij}^{(N+1-k)}.$$

Let us consider the obtained zero-curvature equations (29) in more detail. Let us introduce new, spectral parameter independent matrices, dependent on the variables introduced above

$$L = \sum_{i,j=1}^{d} l_{ij}^{(N+1)} X_{ij} \qquad M_k = \sum_{i,j=1}^{d} m^{(k)}\left(l_{ij}^{(N+1)}, l_{ij}^{(N)}, \ldots, l_{ij}^{(N+1-k)}\right) X_{ij} \qquad k \in 1, l. \tag{30}$$

Direct calculation gives the following statement:

Proposition 8. *Equations (29) are equivalent to the following system of differential equations on the matrices L and M_k, $k \in 1, l$:*

$$\frac{\partial L}{\partial t_l} - \frac{\partial M_l}{\partial x} = [L, M_l]_A \tag{31a}$$

$$\frac{\partial L}{\partial x} = [L, M_1] \tag{31b}$$

$$\frac{\partial M_k}{\partial x} = [L, M_{k+1}] - [L, M_k]_A \qquad \text{where } k \in 1, l-1. \tag{31c}$$

Here $[L, M_k]_A \equiv LAM_k - M_kAL$, and $A = \mathrm{diag}(a_1, a_2, \ldots, a_d)$.

Equations (29) are the family of differential equations in partial derivatives on the matrix elements of the matrices L and M_k.

In order to get rid of the additional dynamical variables we have to express $m^{(k)}\left(l_{ij}^{(N+1)}, l_{ij}^{(N)}, \ldots, l_{ij}^{(N-k)}\right)$ via $l_{ij}^{(N+1)}, \left(l_{ij}^{(N+1)}\right)_x, \left(l_{ij}^{(N+1)}\right)_{xx}, \ldots$ and impose additional constraints on the matrix elements $l_{ij}^{(N+1)}$ of matrix L (see previous subsection). In the result we obtain nonlinear differential equations in partial derivatives on the matrix elements of the matrix L of the following form:

$$\frac{\partial L}{\partial t_l} = \frac{\partial M_l(L, L_x, L_{xx}, \ldots)}{\partial x} + [L, M_l(L, L_x, L_{xx}, \ldots)]_A. \tag{32}$$

We will call them *higher equations of the DHM hierarchy*.

Example of the higher equation of the hierarchy. In the case of general l, higher equations of the DHM hierarchy have a very complicated form. We will now consider only the $l = 2$ example. In this case we have the following set of equations:

$$\frac{\partial L}{\partial t_2} - \frac{\partial M_2}{\partial x} = [L, M_2]_A \tag{33a}$$

$$\frac{\partial M_1}{\partial x} = [L, M_2] - [L, M_1]_A \tag{33b}$$

$$\frac{\partial L}{\partial x} = [L, M_1]. \tag{33c}$$

In order to obtain one equation in the partial derivatives on elements of matrix L we have to express M_1 via L and L_x and M_2 via L, L_x and L_{xx}. Let us consider the explicit form of the resulting equations in the case $\mathfrak{g} = so(3)$. In this case we will again consider elements of $\mathfrak{g}$ as vectors: $\vec{L} = \sum_{k=1}^{3} L_k X_k$, $\vec{M}_1 = \sum_{k=1}^{3} M_{1k} X_k$, $\vec{M}_2 = \sum_{k=1}^{3} M_{2k} X_k$ and rewrite zero-curvature equations (33) as follows:

$$\frac{\partial \vec{L}}{\partial t_2} - \frac{\partial \vec{M}_2}{\partial x} = A[\vec{L} \times \vec{M}_2] \tag{34a}$$

$$\frac{\partial \vec{M}_1}{\partial x} = [\vec{L} \times \vec{M}_2] - A[\vec{L} \times \vec{M}_1] \tag{34b}$$

$$\frac{\partial \vec{L}}{\partial x} = [\vec{L} \times \vec{M}_1]. \tag{34c}$$

Imposing the standard constraint: $\vec{L}^2 = \sum_{i=1}^{3} L_i^2 = 1$, we obtain for $\vec{M}_1$ and $\vec{M}_2$ the following expressions:

$$\vec{M}_1 = -\left[\vec{L} \times \frac{\partial \vec{L}}{\partial x}\right] + c_1(L)\vec{L}$$

$$\vec{M}_2 = -\left[\vec{L} \times \frac{\partial}{\partial x}(\vec{M}_1 + A\vec{L})\right] + c_2(L)\vec{L}$$

where $c_1(L) = \frac{1}{2}(A\vec{L}, \vec{L}) + c_1$ (see previous subsection). In order to obtain the explicit form of the coefficient $c_2(L)$ it is enough to take into consideration that in each sector $\mathcal{M}_N$ matrices M_i and L are expressed via $l_{ij}^{(N+1)}$, $l_{ij}^{(N)}$ and $l_{ij}^{(N-1)}$ and

$$H_{D-2}^2 = (\vec{M}_2, \vec{L}) + (\vec{L}, A\vec{M}_1) + (A\vec{L}, \vec{M}_1) + \tfrac{1}{2}(\vec{M}_1, \vec{M}_1) + (A\vec{L}, A\vec{L})$$

is a Casimir function of the brackets $\{\ ,\ \}_0$, and hence is equal to a constant in $\mathcal{M}_N$. Taking this into account along with the explicit form of $\vec{M}_1$ as a function of $\vec{L}$, $\vec{L}_x$ one can easily derive an explicit expression for $c_2(L) \equiv (\vec{M}_2, \vec{L})$ as a function of $\vec{L}$, $\vec{L}_x$.

The resulting higher DHM equations will have the following complicated form:

$$\frac{\partial \vec{L}}{\partial t_2} = -\frac{\partial^3 \vec{L}}{\partial x^3} - \frac{\partial}{\partial x}\left(c_1(L)\left[\vec{L} \times \frac{\partial \vec{L}}{\partial x}\right] + A\left[\vec{L} \times \frac{\partial \vec{L}}{\partial x}\right] + \left[\vec{L} \times A\frac{\partial \vec{L}}{\partial x}\right]\right)$$
$$+ \frac{\partial}{\partial x}\left(\left(c_2(L) - \left(\frac{\partial \vec{L}}{\partial x}, \frac{\partial \vec{L}}{\partial x}\right)\right)\vec{L} + c_1(L)A\vec{L} + A^2\vec{L}\right). \tag{35}$$

Remark 9. In the rational degeneration ($a_i = 0 \Rightarrow A \equiv 0$) equations (35) acquire the following form:

$$\frac{\partial \vec{L}}{\partial t_2} = -\frac{\partial^3 \vec{L}}{\partial x^3} - \frac{\partial}{\partial x}\left(c_1(L)\left[\vec{L} \times \frac{\partial \vec{L}}{\partial x}\right]\right) + \frac{\partial}{\partial x}\left(\left(c_2(L) - \left(\frac{\partial \vec{L}}{\partial x}, \frac{\partial \vec{L}}{\partial x}\right)\right)\vec{L}\right) \tag{36}$$

where $c_1(L) = c_1, c_2(L) = c_2 - \frac{1}{2}\left(\frac{\partial \vec{L}}{\partial x}, \frac{\partial \vec{L}}{\partial x}\right)$. Equation (36) after substitution $t_2 \rightarrow -t_2$ coincides with the first higher equation of the Heisenberg magnet hierarchy.

4.4. Conclusion and discussion

In the present paper we have constructed new types of integrable nonlinear equations in the partial derivatives using special quasigraded Lie algebras $\tilde{\mathfrak{g}}_{\mathcal{H}}$ on hyperelliptic curves [8, 9]. We have shown that the constructed equations admit zero-curvature representations and possess an infinite sequence of the embedded 'finite-gap' sectors $\mathcal{M}_N$: $\mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_N \subset \cdots$, each of which coincides with an integrable Hamiltonian equation of the Euler–Arnold type.

An interesting problem, concerning the constructed equations in the partial derivatives, is whether they could be interpreted as Hamiltonian equations on the orbits of the centrally extended algebra of periodic functions of variable x with the values in $\tilde{\mathfrak{g}}_{\mathcal{H}}$. There exist some hints (see [7]) that such an interpretation is possible, at least in the case of $\mathfrak{g} = so(3)$. We plan to return to this problem in our subsequent papers.

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